UNIFORM ESTIMATES ON THE FISHER INFORMATION FOR SOLUTIONS TO BOLTZMANN AND LANDAU EQUATIONS

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ABSTRACT. In this note we prove that, under some minimal regularity assumptions on the initial datum, solutions to the spatially homogenous Boltzmann and Landau equations for hard potentials uniformly propagate the Fisher information. The proof of such a result is based upon some explicit pointwise lower bound on solutions to Boltzmann equation and strong diffusion properties for the Landau equation. We include an application of this result related to emergence and propagation of exponential tails for the solution's gradient. These results complement estimates provided in [24, 26, 14, 23].

Keywords. Boltzmann equation, Landau equation, Fisher information, propagation of regularity.

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1. INTRODUCTION

The Fisher information functional was introduced in [17]

$$\mathcal{I}(f) := 4\int_{\mathbb{R}^d} \left|\nabla\sqrt{f(v)}\right|^2 \mathrm{d}v \tag{1.1}$$

as a tool in statistics and information theory. It revealed itself a very powerful tool to control regularity and rate of convergence for solutions to several partial differential equations. In particular, in the study of Fokker-Planck equation, the control of the Fisher information along the Orstein-Uhlenbeck semigroup is the key point for the exponential rate of convergence to equilibrium [12] in relative entropy terms. Variants of such an approach can be applied to deal with more general parabolic problems [13]. For these kind of problems, the Fisher information turns out to play the role of a Lyapunov functional.

Such techniques have also been applied in the context of general collisional kinetic equation. In particular, for the Boltzmann equation with Maxwell molecules, exploiting commutations between the Boltzmann collision operator and the Orstein-Uhlenbeck semigroup, the Fisher information serves as a Lyapunov functional for the study of the long time relaxation [22, 10]. In [8, 9, 26], the Fisher information was applied for general collision kernels in relation to the entropy production bounds for the Boltzmann equation. Later in [27], ground breaking work related to the Cercignani's conjecture was made using the Fisher information and the ideas preceding such work. We aim however to emphasize that the present contribution, together with [25, 26], is to our knowledge the only one dealing with the question of uniform-in-time estimates for the Fisher information in the kinetic context.

The aim of the present contribution is to further investigate the properties of Fisher information along solutions to two important kinetic equations: the Boltzmann equation for hard potentials, under cut-off assumption, and the Landau equation for hard potentials. More specifically, we show here that, along solutions to Boltzmann or Landau equations for hard potentials, the Fisher information will remain uniformly bounded

$$\sup_{t\geqslant 0}\mathcal{I}(f(t)) \leqslant C(f_0) < \infty \tag{1.2}$$

under minimal regularity assumptions on the initial datum. The minimality is understood in terms of smoothness required for the initial datum and not in terms of number of moments, however, we have tried to be as frugal as possible in this latter issue. In remarks below Theorem 1.1 and Theorem 1.2 we expand on the interpretation of the results and the way they are, or are not, optimal.

For the Boltzmann equation estimate (1.2) improves, under less restrictive conditions in the model, the local in time estimate obtained in [26] which reads

$$\mathcal{I}(f(t)) \leqslant e^{c\,t}\big(2I(f_0) + c\,(1+t^3)\big)\,, \quad \text{for some explicit } \; c>0\,.$$

This bound was obtained in the context of Maxwell molecule type of models and does not directly apply to the case of hard potentials, with exception of hard spheres, that we treat here. Interestingly, the bound (1.2) can be used to generalise, to the context of hard potential models, estimates that were designed for a Maxwell gas with respect to propagation of smoothness such as in [10]. Let us mention that for the case of Maxwell model, the Fisher information is in fact non-increasing for both the Boltzmann and Landau equations, see [21, 26, 25, 15]. This explains why the Fisher information is used to prove exponential relaxation towards thermodynamical equilibrium in this case. Furthermore, the Maxwell model can be compared to other models as well to obtain algebraic rate of relaxation towards equilibrium.

As an application of the uniform propagation of the Fisher information, one can deduce that, for any $t_0>0$,

$$\sup_{t\geqslant t_0>0}\int_{\mathbb{R}^d}\big|\nabla f(t,v)\big|e^{c\,|v|^{\gamma}}\mathrm{d}v \leqslant C(f_0,t_0)<\infty\,, \quad \text{for some explicit } \; c>0\,,$$

in a relatively simple manner (relatively to [5] for example). The techniques to prove the bound (1.2) seem to differ in nature for the study of Boltzmann (with cutoff) and Landau equations. The reason for this difference is that while Landau's equation is strongly diffusive, the Boltzmann equation is weakly diffusive. For the Boltzmann equation, we exploit the *appearance of pointwise exponential lower bounds* for solutions obtained in [20] whereas, for the Landau equation, we use the instantaneous regularizing effect to control, for time $t\geqslant t_0>0$ the Fisher information by Sobolev regularity bounds while, for small time $0<t<t_0$, the Fisher information is controlled thanks to a *new energy estimate* for solutions to the Landau equation. These proofs are related in the sense that for positive functions, the models' solutions, regularity imply a particular behaviour near to zero.

1.1. **Notations.** Let us introduce some useful notations for function spaces. For any $p\geqslant 1$ and $q\geqslant 0$, we define the space $L^p_q(\mathbb{R}^d)$ through the norm

$$\|f\|_{L^p_q} := \left(\int_{\mathbb{R}^d}|f(v)|^p\langle v\rangle^{pq}\mathrm{d}v\right)^{1/p}\,,$$

i.e. $L^p_q(\mathbb{R}^d)=\{f\;:\mathbb{R}^d\to\mathbb{R}\;;\;\|f\|_{L^p_q}<\infty\}$ where, for $v\in\mathbb{R}^d$, $\langle v\rangle=\sqrt{1+|v|^2}$. We also define, for $k\geqslant 0$,

$$H^k_q(\mathbb{R}^d)=\left\{f\in L^2(\mathbb{R}^d)\;;\;(1-\Delta)^{\frac{k}{2}}f\in L^2_q(\mathbb{R}^d)\right\}\,,$$

where the operator $(1-\Delta)^{\frac{k}{2}}$ is defined through its Fourier transform

$$\mathcal{F}\big\{(1-\Delta)^{\frac{k}{2}}f\big\}(\xi)=\langle\xi\rangle^k\mathcal{F}\{f\}(\xi)=:\langle\xi\rangle^k\int_{\mathbb{R}^d}e^{ix\cdot\xi}f(x)\mathrm{d}x\,,\quad \xi\in\mathbb{R}^d.$$

When we write $H^{k^+}_q(\mathbb{R}^d)$, for some $k\in\mathbb{R}$, we simply mean that the positive part $k^+:=\max\{0,k\}$ of k is taken. Also, we define $L^1_{\log}(\mathbb{R}^d)$ as

$$L^1_{\log}(\mathbb{R}^d)=\left\{f\in L^1(\mathbb{R}^d)\;;\;\int_{\mathbb{R}^d}|f(v)|\,|\log(|f(v)|)|\,\mathrm{d}v<\infty\right\}.$$

1.2. **The Boltzmann equation.** Let us now enter into the details by considering the solution $f(t,v)$ to the Boltzmann equation

$$\partial_t f(t,v) = \mathcal{Q}(f,f)(t,v)\,, \qquad v \in \mathbb{R}^d\,. \tag{1.3}$$

We consider kernels satisfying $\|b\|_{L^1(\mathbb{S}^{d-1})} < \infty$, thus, it is possible to write the collision operator in gain and loss operators

$$\mathcal{Q}(f,g) = \mathcal{Q}^+(f,g) - g\,\mathcal{R}(f)\,,$$

where the collision operator is given by

$$\begin{aligned}\mathcal{Q}^+(f,g)(v) &:= \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} b(\cos\theta)|v-v_\star|^\gamma\, f(v'_\star)\, g(v')\mathrm{d}v_\star\mathrm{d}\sigma\,,\\ \mathcal{R}(f)(v) &:= \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} b(\cos\theta)|v-v_\star|^\gamma\, f(v_\star)\mathrm{d}v_\star\mathrm{d}\sigma = \|b\|_{L^1(\mathbb{S}^{d-1})}\big(f * |u|^\gamma\big)(v)\,.\end{aligned}$$

with $\cos\theta = \frac{v-v_\star}{|v-v_\star|}\cdot\sigma$ and

$$v' = \frac{v+v_\star}{2} + \frac{|v-v_\star|}{2}\,\sigma, \qquad v'_\star = \frac{v+v_\star}{2} - \frac{|v-v_\star|}{2}\,\sigma, \qquad \sigma\in\mathbb{S}^{d-1}.$$

We will consider hard potentials $\gamma\in(0,1]$. Also, for technical simplicity, we restrict ourselves to $d \geqslant 3$.

Theorem 1.1. ***(Uniform propagation of the Fisher information)*** *Let $b \in L^2(\mathbb{S}^{d-1})$ be the angular scattering kernel, $d\geqslant 3$ and $\gamma\in(0,1]$. Assume also that the initial datum $f_0 \geqslant 0$ satisfies*

$$f_0 \in L^1_\eta(\mathbb{R}^d)\cap L^2_\mu(\mathbb{R}^d)\cap H_\nu^{\frac{(5-d)^+}{2}}(\mathbb{R}^d),$$

for some $\nu > 3+\gamma+\frac{d}{2}$, $\mu\geqslant \nu+1+\frac{\gamma}{2}$, $\eta\geqslant\mu+d$ and

$$\int_{\mathbb{R}^d} f_0(v)\,v\,\mathrm{d}v = 0, \qquad\qquad \mathcal{I}(f_0)<\infty.$$

Then, the unique solution $f(t)=f(t,v)\geqslant 0$ to (1.3) satisfies

$$\sup_{t\geqslant 0}\mathcal{I}(f(t))\leqslant C\,,$$

for some positive constant C depending on $\mathcal{I}(f_0)$ and the $L^1_\eta\cap L^2_\mu\cap H_\nu^{\frac{(5-d)^+}{2}}$-norm of f_0.

Remark 1.1. *Let us explain the optimality of Theorem 1.1. The condition $f_0\in L^1_2$ is needed for the well–posedness of the Cauchy problem for the equation. Clearly, for propagation of the Fisher information, $\mathcal{I}(f_0)$ must be finite. When $d\geqslant 5$ there is no need for gradient regularity and the Theorem holds true for*

$$f_0\in L^1_\eta(\mathbb{R}^d)\cap L^2_\mu(\mathbb{R}^d) \qquad \text{for any } \mu\geqslant 4+\frac{3}{2}\gamma+\frac{d}{2} \text{ and } \eta\geqslant\mu+d.$$

This is optimal with respect to regularity required for f_0. When $d=3$ or $d=4$, there is a gradient requirement. Generally speaking the Fisher information and gradient integrability are not comparable objects, thus, for these cases it is unclear if such condition on the gradient is a technical artefact or real. However, if f_0 is assumed to be bounded, one can use the estimate

$$\|\nabla f_0\|^2_{L^2_\nu(\mathbb{R}^d)}\leqslant \|f_0\|_{L^\infty_{2\nu}(\mathbb{R}^d)}\,\mathcal{I}(f_0)\,, \qquad \nu\geqslant 0\,,$$

and classical regularity theory for the Boltzmann equation, for instance [19], to get rid of the gradient assumption.

Remark 1.2. *If the reader is willing to accept more regularity in the initial data, say $f_0\in H^2_\nu(\mathbb{R}^d)$ for some $\nu>\frac{d}{2}$, then Theorem 1.1 remains valid for $b\in L^1(\mathbb{S}^{d-1})$ using the propagation of regularity given in [5] and the control of the Fisher information using the $H^2_\nu(\mathbb{R}^d)$ norm, see [23, Lemma 1].*

1.3. **The Landau equation.** As mentioned earlier, we also investigate the case of solutions to the homogeneous Landau equation. Recall that such an equation reads

$$\partial_t f = \mathcal{Q}_L(f,f)\,, \qquad v \in \mathbb{R}^d\,. \tag{1.4}$$

The collision operator is defined as

$$\mathcal{Q}_L(f,f)(v) = \nabla \cdot \int_{\mathbb{R}^d} A(v-v_\star)\Big(f(v_\star)\nabla f(v) - f(v)\nabla f(v_\star)\Big)\mathrm{d}v_\star \tag{1.5}$$

where the matrix $A(z) = (A_{ij}(z))_{i,j=1,\ldots,d}$ is given by

$$A_{ij}(z) = \left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right)\Phi(z), \qquad \Phi(z) := |z|^{2+\gamma}\,.$$

We concentrate the study in the hard potential case $\gamma \in (0,1]$. We refer to [14] for a methodical study of the Landau equation in this setting. The Landau equation can be written in the form of a nonlinear parabolic equation:

$$\partial_t f(t,v) - \nabla \cdot \big(a(v)\nabla f(t,v) - b(v)\,f(t,v)\big) = 0\,, \tag{1.6}$$

where the matrix $a(v)$ and the vector $b(v)$ are given by

$$a := A * f\,, \qquad b := \nabla \cdot A * f\,.$$

The minimal conditions that will be required on the initial datum f_0 are finite mass, energy and entropy

$$m_0 := \int_{\mathbb{R}^d} f_0(v)\mathrm{d}v < +\infty\,, \quad E_0 := \int_{\mathbb{R}^d} |v|^2 f_0(v)\mathrm{d}v < +\infty\,, \quad H_0 := \int_{\mathbb{R}^d} f_0(v)\log f_0(v)\mathrm{d}v < +\infty\,.$$

For technical reasons, to assure conservation of energy, a moment higher than 2 is assumed as well. In this situation, [14, Proposition 4] asserts that the equation is uniformly elliptic, that is,

$$a(v)\xi \cdot \xi \geqslant a_0 \langle v\rangle^\gamma |\xi|^2\,, \qquad \forall\, v \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d$$

for some positive constant $a_0 := a_0(m_0, E_0, H_0)$. Under these assumptions, the Cauchy theory, including infinite regularization and moment propagation, has been developed in [14, 15]. As in the Boltzmann case, the Fisher information have been used for the analysis of convergence towards equilibrium, see for instance [15, 23, 24], and also for analysis of regularity, see [16]. Concerning regularisation, the idea is to establish an inequality of the form

$$\int_{\mathbb{R}^d} |\nabla\sqrt{f}|^2 \mathrm{d}v \leqslant C\big(\mathcal{D}(f) + 1\big),$$

with constant C depending only on m_0, E_0, H_0, which are the physical conserved quantities, and where $\mathcal{D}(f)$ denotes the entropy production associated to $\mathcal{Q}_L$, i.e.

$$\mathcal{D}(f) = -\int_{\mathbb{R}^d} \mathcal{Q}_L(f,f)\log f \mathrm{d}v.$$

Since, along solutions to the Landau equation $f(t) = f(t,v)$ it holds that

$$0 \leqslant \int_0^t \mathcal{D}(f(s))\mathrm{d}s \leqslant C_{\mathcal{D}}(m_0, E_0, H_0, t)\,,$$

such inequality leads to estimate on the *time integrated* Fisher information. Then, one uses Sobolev inequality to obtain control on the entropy or a higher norm.

For the Fisher information itself, at least for the hard potential case, the following result follows. The theorem is stated for $d = 3$ because it uses several results given in [14].

Theorem 1.2. *Assume that the initial datum $f_0 \geqslant 0$ has finite mass m_0, energy E_0 and entropy H_0 and satisfies in addition*

$$\int_{\mathbb{R}^3} \langle v \rangle^2 f_0(v) \log f_0(v) \mathrm{d}v < +\infty\,, \qquad \int_{\mathbb{R}^3} \langle v \rangle^{2+\gamma+\epsilon} f_0(v) \mathrm{d}v < +\infty\,, \tag{1.7}$$

for some $\epsilon > 0$. Assume moreover that $\mathcal{I}(f_0) < \infty$. Then, there exists a weak solution $f(t) = f(t,v)$ to (1.4) with initial datum f_0 satisfying

$$\sup_{t \geqslant 0} \mathcal{I}(f(t)) \leqslant C_F^0\,,$$

where the constant C_F^0 depends on m_0, E_0, H_0, the quantities in (1.7), and the initial Fisher information.

Remark 1.3. *A condition for well–posedness and regularisation of the Cauchy problem for the Landau equation is $f_0 \in L^2_s$, with $s > (5\gamma + 15)/2$, see* [14, Theorem7]. *Thus, the assumptions on f_0 in Theorem 1.2 are quite general. Since $\mathcal{I}(f_0) < \infty$ is necessary for uniform propagation of the Fisher information, Theorem 1.2 is optimal with respect to the regularity required for f_0. Furthermore, inspecting the results for existence and regularity of solutions given in* [14]*, the requirement on the moments for f_0 in (1.7) appears very close to optimal.*

The rest of the document is divided in three sections, Section 2 is devoted to the proof of Theorem 1.1 and Section 3 is concerned with the proof of Theorem 1.2. The final section is an Appendix where the reader will find helpful facts about Boltzman (Appendix A.) and Landau (Appendix B.) equations that will be needed along the arguments.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we consider in all this section a solution $f(t) = f(t,v)$ to the Boltzmann equation (1.3) that conserves mass, momentum, and energy. One has first the following lemma.

Lemma 2.1. *The Fisher information of $f(t,\cdot)$ satisfies*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(f(t)) = &-2\int_{\mathbb{R}^d} \log f(t,v)\ \Delta_v \mathcal{Q}^+(f,f)(t,v)\mathrm{d}v - 4\int_{\mathbb{R}^d} \left|\nabla\sqrt{f(t,v)}\right|^2 \mathcal{R}(f)(t,v)\mathrm{d}v \\ &- 2\int_{\mathbb{R}^d} \nabla f(t,v)\cdot\nabla\mathcal{R}(f)(t,v)\,\mathrm{d}v - \int_{\mathbb{R}^d} |\nabla \log f(t,v)|^2\, \mathcal{Q}^+(f,f)(t,v)\mathrm{d}v\,. \end{aligned} \tag{2.1}$$

Proof. One first notices that $g_i(t,v) := \partial_{v_i}\sqrt{f(t,v)}$ satisfies

$$\begin{aligned} \partial_t g_i(t,v) = \partial_{v_i}\left(\frac{1}{2\sqrt{f(t,v)}}\mathcal{Q}(f,f)(t,v)\right) &= -\frac{1}{2f(t,v)} g_i(t,v)\ \mathcal{Q}(f,f)(t,v) + \frac{1}{2\sqrt{f(t,v)}}\partial_{v_i}\mathcal{Q}(f,f)(t,v) \\ &= -\frac{1}{2f(t,v)} g_i(t,v)\ \mathcal{Q}^+(f,f)(t,v) + \frac{1}{2}g_i(t,v)\mathcal{R}(f)(t,v) \\ &\qquad + \frac{1}{2\sqrt{f(t,v)}}\partial_{v_i}\mathcal{Q}^+(f,f)(t,v) - \frac{1}{2\sqrt{f(t,v)}}\partial_{v_i}\left(f(t,v)\mathcal{R}(f)(t,v)\right). \end{aligned}$$

Multiplying by $g_i(t,v)$ and integrating over $\mathbb{R}^d$ we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g_i(t)\|^2_{L^2} = &-\frac{1}{2}\int_{\mathbb{R}^d} \frac{g_i^2(t,v)}{f(t,v)}\ \mathcal{Q}^+(f,f)(t,v)\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^d} g_i^2(t,v)\mathcal{R}(f)(t,v)\mathrm{d}v \\ &+ \frac{1}{2}\int_{\mathbb{R}^d} \frac{g_i(t,v)}{\sqrt{f(t,v)}}\partial_{v_i}\mathcal{Q}^+(f,f)(t,v)\mathrm{d}v - \frac{1}{2}\int_{\mathbb{R}^d} \frac{g_i(t,v)}{\sqrt{f(t,v)}}\partial_{v_i}\left(f(t,v)\mathcal{R}(f)(t,v)\right)\mathrm{d}v. \end{aligned}$$

Noticing that

$$\frac{g_i^2(t,v)}{f(t,v)} = \left(\frac{\partial_{v_i} f(t,v)}{2f(t,v)}\right)^2 = \frac{1}{4}\left(\partial_{v_i}\log f(t,v)\right)^2\,,$$

and

$$\frac{g_i(t,v)}{\sqrt{f(t,v)}}\partial_{v_i}\left(f(t,v)\mathcal{R}(f)(t,v)\right) = \frac{g_i(t,v)\partial_{v_i}f(t,v)}{\sqrt{f(t,v)}}\mathcal{R}(f)(t,v) + g_i(t,v)\sqrt{f(t,v)}\partial_{v_i}\mathcal{R}(f)(t,v)$$
$$= \frac{(\partial_{v_i}f(t,v))^2}{2f(t,v)}\mathcal{R}(f)(t,v) + \frac{1}{2}\partial_{v_i}f(t,v)\partial_{v_i}\mathcal{R}(f)(t,v)\,,$$

we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g_i(t)\|_{L^2}^2 = -\frac{1}{8}\int_{\mathbb{R}^d}\left(\partial_{v_i}\log f(t,v)\right)^2\,\mathcal{Q}^+(f,f)(t,v)\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^d} g_i^2(t,v)\mathcal{R}(f)(t,v)\mathrm{d}v$$
$$+\frac{1}{4}\int_{\mathbb{R}^d}\partial_{v_i}\log f(t,v)\,\partial_{v_i}\mathcal{Q}^+(f,f)(t,v)\mathrm{d}v - \frac{1}{4}\int_{\mathbb{R}^d}\frac{(\partial_{v_i}f(t,v))^2}{f(t,v)}\mathcal{R}(f)(t,v)\mathrm{d}v$$
$$-\frac{1}{4}\int_{\mathbb{R}^d}\partial_{v_i}f(t,v)\partial_{v_i}\mathcal{R}(f)(t,v)\mathrm{d}v\,.$$

Using an integration by part in the third integral, and since $\dfrac{(\partial_{v_i}f(t,v))^2}{4f(t,v)} = g_i^2(t,v)$, this results easily in

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g_i(t)\|_{L^2}^2 = -\frac{1}{4}\int_{\mathbb{R}^d}\left(\partial_{v_i}\log f(t,v)\right)^2\,\mathcal{Q}^+(f,f)(t,v)\mathrm{d}v - \int_{\mathbb{R}^d} g_i^2(t,v)\mathcal{R}(f)(t,v)\mathrm{d}v$$
$$-\frac{1}{2}\int_{\mathbb{R}^d}\log f(t,v)\,\partial^2_{v_iv_i}\mathcal{Q}^+(f,f)(t,v)\mathrm{d}v - \frac{1}{2}\int_{\mathbb{R}^d}\partial_{v_i}f(t,v)\partial_{v_i}\mathcal{R}(f)(t,v)\mathrm{d}v$$

which yields the desired result after adding in $i = 1, 2, \ldots, d$. □

All terms in (2.1) are relatively easy to estimate with exception, perhaps, of the term involving $\Delta Q^+(f,f)$. This is the step where the instantaneous appearance of a lower gaussian barrier is important, in particular, for the estimation of the constant $c_\varepsilon(t)$ in the following lemma.

Lemma 2.2. *Let $f(t)\geqslant 0$ be a sufficiently smooth solution of the Boltzmann equation. Then, for any $\varepsilon > 0$*

$$\int_{\mathbb{R}^d}\big|\log f(t,v)\big|\,\big|\Delta_v\mathcal{Q}^+(f,f)(t,v)\big|\mathrm{d}v \leqslant C(\varepsilon,d)\Big(c_\varepsilon(t) + \|f(t)\|_{L^2}\Big)\Big(\|f(t)\|^2_{H^s_{\eta_1}} + \|f(t)\|^2_{L^1_{\eta_2}}\Big)\,,$$

where $c_\varepsilon(t) := C_\varepsilon\big(1+\log^+(1/t)\big)$ for some universal constant $C_\varepsilon > 0$, and

$$\eta_1 := \frac{6+2\gamma+d+3\,\varepsilon}{2}, \qquad \eta_2 := \frac{4+2\gamma+d+3\,\varepsilon}{2}, \qquad s = \frac{(5-d)^+}{2}\leqslant 1\,.$$

Proof. Using Theorem A.1, we get that

$$\int_{\mathbb{R}^d}\big|\log f(t,v)\big|\,\big|\Delta_v\mathcal{Q}^+(f,f)(t,v)\big|\mathrm{d}v \leqslant \int_{\mathbb{R}^d}\Big(c_\varepsilon(t)\langle v\rangle^{2+\varepsilon} + f(t,v)\Big)\,\big|\Delta_v\mathcal{Q}^+(f,f)(t,v)\big|\mathrm{d}v\,.$$

Thus,

$$\int_{\mathbb{R}^d}\big|\log f(t,v)\big|\,\big|\Delta_v\mathcal{Q}^+(f,f)(t,v)\big|\mathrm{d}v \leqslant c_\varepsilon(t)\,\|\Delta_v\mathcal{Q}^+(f(t),f(t))\|_{L^1_{2+\varepsilon}} + \|f(t)\|_{L^2}\|\mathcal{Q}^+(f(t),f(t))\|_{H^2}.$$

Using the interpolation

$$\|h\|_{L^1_s}\leqslant C_\tau(d)\|h\|_{L^2_{s+\tau}} \qquad \forall\,\tau > d/2, \quad s\in\mathbb{R}$$

for constant $C_\tau(d) = \|\langle\cdot\rangle^{-\tau}\|_{L^2}$, we get that for $\tau = \frac{d+\varepsilon}{2}$,

$$\|\Delta_v\mathcal{Q}^+(f(t),f(t))\|_{L^1_{2+\varepsilon}}\leqslant C_{\frac{d+\varepsilon}{2}}(d)\|\mathcal{Q}^+(f(t),f(t))\|_{H^2_{2+\frac{3\varepsilon+d}{2}}}\,.$$

This results in

$$\int_{\mathbb{R}^d}\big|\log f(t,v)\big|\,\big|\Delta_v\mathcal{Q}^+(f,f)(t,v)\big|\mathrm{d}v \leqslant C_{\frac{d+\varepsilon}{2}}(d)\Big(c_\varepsilon(t) + \|f(t)\|_{L^2}\Big)\|\mathcal{Q}^+(f(t),f(t))\|_{H^2_{2+\frac{3\varepsilon+d}{2}}}\,.$$

Now, using Theorem A.4 we can estimate the last term and get

$$\int_{\mathbb{R}^d} \big|\log f(t,v)\big|\,\big|\Delta_v \mathcal{Q}^+(f,f)(t,v)\big|\mathrm{d}v \leqslant C(\varepsilon,d)\Big(c_\varepsilon(t)+\|f(t)\|_{L^2}\Big)\Big(\|f(t)\|^2_{H^s_{\eta_1}}+\|f(t)\|^2_{L^1_{\eta_2}}\Big) \tag{2.2}$$

with η_1, η_2 and s as defined in the statement of the lemma. □

Proof of Theorem 1.1. We start with (2.1) and neglect the nonpositive last term in the right side. It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(f(t)) \leqslant{}& -2\int_{\mathbb{R}^d}\log f(t,v)\ \Delta_v\mathcal{Q}^+(f,f)(t,v)\mathrm{d}v\\ &-4\int_{\mathbb{R}^d}\Big|\nabla\sqrt{f(t,v)}\Big|^2\mathcal{R}(f)(t,v)\mathrm{d}v-2\int_{\mathbb{R}^d}\nabla f(t,v)\cdot\nabla\mathcal{R}(f)(t,v)\,\mathrm{d}v\,.\end{aligned}$$

Additionally, thanks to (A.1), one has $\mathcal{R}(f)(v)\geqslant\kappa_0\langle v\rangle^\gamma$. And due to integration by parts and (A.2)

$$\begin{aligned}-2\int_{\mathbb{R}^d}\nabla f(t,v)\cdot\nabla\mathcal{R}(f)(t,v)\,\mathrm{d}v &= 2\int_{\mathbb{R}^d} f(t,v)\,\Delta_v\mathcal{R}(f)(t,v)\mathrm{d}v\\ &\leqslant C_{d,\gamma}\|b\|_{L^1(\mathbb{S}^{d-1})}\|f\|_{L^1}\Big(\|f\|_{L^1}+\|f\|_{H^{\frac{(4-d)^+}{2}}}\Big)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(f(t))+\kappa_0\,\mathcal{I}(f(t)) &\leqslant 2\int_{\mathbb{R}^d}\big|\log f(t,v)\big|\,\big|\Delta_v\mathcal{Q}^+(f,f)(t,v)\big|\mathrm{d}v+2\int_{\mathbb{R}^d}f(t,v)\,\Delta_v\mathcal{R}(f)(t,v)\mathrm{d}v\\ &\leqslant C(\varepsilon,d,b)\Big(c_\varepsilon(t)+\|f(t)\|_{L^2}+\|f(t)\|_{L^1}\Big)\Big(\|f(t)\|^2_{H^s_{\eta_1}}+\|f(t)\|^2_{L^1_{\eta_2}}+1\Big)\,,\end{aligned}$$

where we used, in addition to previous estimates, Lemma 2.2 for the second inequality. Here η_1, η_2, and s are those defined in such lemma.

Under our assumptions on f_0 and for a suitable choice of $\varepsilon>0$ small enough, the $L^1_{\eta_2}$ and $H^1_{\eta_1}$ norms of $f(t)$ are uniformly bounded, see Theorems A.2 and A.5. Thus, we obtain that, for such choice of $\varepsilon>0$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(f(t))+\kappa_0\,\mathcal{I}(f(t))\leqslant C(f_0)(1+\log^+(1/t)),\qquad t>0\,.$$

Using that the mapping $t\mapsto 1+\log^+(1/t)$ is integrable at $t=0$, a direct integration of this differential inequality implies that $\sup_{t\geqslant0}\mathcal{I}(f(t))\leqslant\mathcal{I}(f_0)+C(f_0)<\infty$. This proves the result. □

A consequence of this result is the exponentially weighted generation/propagation of the solution's gradient. Indeed, one knows thanks to [2] that $\|f(t)e^{c\min\{1,t\}|v|^\gamma}\|_{L^1}\leqslant C(f_0)$ for some sufficiently small $c>0$ and constant $C(f_0)$ depending only on mass and energy. Then,

$$\begin{aligned}\int_{\mathbb{R}^d}\big|\nabla f(t,v)\big|e^{\frac{c}{2}\min\{1,t\}|v|^\gamma}\mathrm{d}v &= 2\int_{\mathbb{R}^d}\big|\nabla\sqrt{f}\big|\sqrt{f}\,e^{\frac{c}{2}\min\{1,t\}|v|^\gamma}\mathrm{d}v\\ &\leqslant\mathcal{I}(f(t))^{\frac12}\big\|f(t)e^{c\min\{1,t\}|v|^\gamma}\big\|^{\frac12}_{L^1}\leqslant C(f_0)\sqrt{\mathcal{I}(f(t))}.\end{aligned}$$

This proves that exponential moments of the gradient $\nabla f(t,v)$ are uniformly bounded by some positive constant depending only on the initial datum f_0.

3. Proof of Theorem 1.2

In this section, we prove the uniform in time estimate on the Fisher information for solutions to the Landau equation. The strong diffusion properties of Landau make the Fisher information more suited to this equation than to Boltzmann.

We assume in all this section that $f(t)=f(t,v)$ is a solution to (1.5) with initial datum $f_0(v)$ with mass m_0, energy E_0. We also assume that f_0 has finite entropy H_0. We shall exploit the parabolic form of the Landau equation that we recall here again

$$\partial_t f - \nabla\cdot(a\nabla f) + \nabla\cdot(bf) = 0\,, \tag{3.1}$$

for $a:=a(v)$ symmetric positive definite matrix and $b:=b(v)$ vector. Recall that, according to (B.1), the matrix $a=a(t,v)$ is uniformly elliptic, i.e.

$$a(t,v)\xi\cdot\xi \geqslant a_0\langle v\rangle^{\gamma}|\xi|^2, \qquad \forall\, v\in\mathbb{R}^3,\ \xi\in\mathbb{R}^3,\ t\geqslant 0\,.$$

Multiplying the equation by $\log f$ and integrating

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f\log f\mathrm{d}v + \int_{\mathbb{R}^3} a\nabla f\cdot\frac{\nabla f}{f}\mathrm{d}v + \int_{\mathbb{R}^3}(\nabla\cdot b)\,f\mathrm{d}v = 0\,.$$

We recall, see (B.2), that

$$\big|(\nabla\cdot b)(v)\big| \leqslant B(m_0,E_0)\langle v\rangle^{\gamma}\,,$$

and, using (B.1)

$$\int_{\mathbb{R}^3} a\nabla f\cdot\frac{\nabla f}{f}\mathrm{d}v \geqslant a_0\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\,\nabla f\cdot\frac{\nabla f}{f}\mathrm{d}v = 4a_0\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\left|\nabla\sqrt{f}\right|^2\mathrm{d}v.$$

As a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f\log f\mathrm{d}v + 4a_0\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\left|\nabla\sqrt{f}\right|^2\mathrm{d}v \leqslant \tilde{B}(m_0,E_0)\,.$$

Integrating in time

$$4a_0\int_0^t\mathrm{d}s\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\left|\nabla\sqrt{f(s,v)}\right|^2\mathrm{d}v \leqslant \int_{\mathbb{R}^3} f_0\log f_0\mathrm{d}v - \int_{\mathbb{R}^3} f(t,v)\log f(t,v)\mathrm{d}v + t\,\tilde{B}(m_0,E_0)\,, \quad t>0\,.$$

Since

$$\sup_{t\geqslant 0}\left|\int_{\mathbb{R}^3} f(t,v)\log f(t,v)\mathrm{d}v\right| \leqslant H(m_0,E_0,H_0)\,,$$

we just proved the first part of the following proposition.

Proposition 3.1. *For a solution $f(t)=f(t,v)$ to the Landau equation one has*

$$4\int_0^t\mathrm{d}s\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\left|\nabla\sqrt{f(s,v)}\right|^2\mathrm{d}v \leqslant C(m_0,E_0,H_0)\big(1+t\big), \qquad t>0\,. \tag{3.2}$$

Moreover, given $k>0$ and $\epsilon>0$, if we assume the initial datum f_0 to be such that

$$\int_{\mathbb{R}^3}\langle v\rangle^{k} f_0(v)\log f_0(v)\mathrm{d}v < +\infty\,, \qquad \int_{\mathbb{R}^3}\langle v\rangle^{k+\gamma+\epsilon} f_0(v)\mathrm{d}v < +\infty\,, \tag{3.3}$$

then

$$4\int_0^t\mathrm{d}s\int_{\mathbb{R}^3}\langle v\rangle^{k+\gamma}\left|\nabla\sqrt{f(s,v)}\right|^2\mathrm{d}v \leqslant C_k(m_0,E_0,H_0)\big(1+t\big), \qquad t>0\,, \tag{3.4}$$

for some positive constant C_k depending on the mass m_0, the energy E_0, the entropy H_0 and the quantities (3.3).

Proof. We already proved (3.2), it remains to prove statement (3.4). For this, we multiply (3.1) by $\langle\cdot\rangle^{\gamma}\log f(t,\cdot)$ and, integrating over $\mathbb{R}^3$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^{k} f(t,v)\log f(t,v)\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)\langle v\rangle^{k}\mathrm{d}v - \int_{\mathbb{R}^3}\langle v\rangle^{k}\nabla\cdot\big(b(v)f(t,v)\big)\log f(t,v)\mathrm{d}v$$
$$+\int_{\mathbb{R}^3}\langle v\rangle^{k}\nabla\cdot\big(a(v)\nabla f(t,v)\big)\log f(t,v)\mathrm{d}v.$$

Note that integrations by parts lead to

$$\int_{\mathbb{R}^3} -\nabla\cdot(a\nabla f)\,\langle v\rangle^k \log f \mathrm{d}v = \int_{\mathbb{R}^3}\langle v\rangle^k a\nabla f\cdot\frac{\nabla f}{f}\mathrm{d}v - k\int_{\mathbb{R}^3}(f\log f - f)\nabla\cdot\big(a\langle v\rangle^{k-2}v\big)\mathrm{d}v$$
$$\geqslant 4a_0\int_{\mathbb{R}^3}\langle v\rangle^{k+\gamma}\big|\nabla\sqrt{f}\big|^2\mathrm{d}v - A_0\,k\int_{\mathbb{R}^3}\langle v\rangle^{k+\gamma}\big(f|\log f| + f\big)\mathrm{d}v\,.$$

The latter inequality follows by using (B.1) and the fact that

$$\big|\nabla\cdot\big(a\langle v\rangle^{k-2}v\big)\big| \leqslant A_0\langle v\rangle^{k+\gamma}\,.$$

Similarly,

$$\left|\int_{\mathbb{R}^3}\nabla\cdot(bf)\,\langle v\rangle^k\log f\,\mathrm{d}v\right| = \left|\int_{\mathbb{R}^3}\nabla\cdot\big(\langle v\rangle^k b\big)\,f\,\mathrm{d}v - k\int_{\mathbb{R}^3}(f\log f)\,b\cdot\langle v\rangle^{k-2}v\,\mathrm{d}v\right|$$
$$\leqslant B_0\int_{\mathbb{R}^3}\langle v\rangle^{k+\gamma}f\,\mathrm{d}v + B_0\,k\int_{\mathbb{R}^3}\langle v\rangle^{k+\gamma}f|\log f|\,\mathrm{d}v.$$

We control the integral with $f|\log f|$ using Lemma B.4 with $\delta>0$ small enough. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^k f(t,v)\,\log f(t,v)\mathrm{d}v + 2a_0\|\langle v\rangle^{\frac{k+\gamma}{2}}\nabla\sqrt{f(t,v)}\|^2_{L^2} \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^k f(t,v)\mathrm{d}v + \tilde{C}_k\,. \tag{3.5}$$

for some positive constant $\tilde{C}_k$ depending only on $\sup_{t\geqslant 0}\|f(t)\|_{L^1_{k+\gamma+\varepsilon}}$ for some arbitrary $\varepsilon>0$. Integrating between 0 and t the previous equation, we get

$$\int_{\mathbb{R}^3}\langle v\rangle^k f(t,v)\,\log f(t,v)\mathrm{d}v + 2a_0\int_0^t\|\langle v\rangle^{\frac{k+\gamma}{2}}\nabla\sqrt{f(s,v)}\|^2_{L^2}\,\mathrm{d}s$$
$$\leqslant \int_{\mathbb{R}^3}\langle v\rangle^k f_0(v)\,\log f_0(v)\mathrm{d}v + \int_{\mathbb{R}^3}\langle v\rangle^k f(t,v)\mathrm{d}v + \tilde{C}_k t\,.$$

The first integral in the left-hand side has no sign but it can be handled thanks to (B.3). The result follows from here using propagation of the moment $k+\gamma+\varepsilon$. □

One notices that, for solutions of the Landau equation for hard potentials, the Fisher information emerges as soon as $t>0$. This result immediately follows from the following lemma.

Lemma 3.1. *Let $f(t)$ be the weak solution to (1.4) with initial datum $f_0\in L^1_{2+\delta}\cap L^1_{\log}(\mathbb{R}^3)$ for some $\delta>0$. For any $t_0>0$, there is $C_{t_0}>0$ depending only on m_0, E_0 and H_0 such that*

$$\sup_{t\geqslant t_0}\mathcal{I}(f(t)) \leqslant C_{t_0}.$$

Proof. The result is a direct consequence of the following link between the Fisher entropy and weighted Sobolev norm, see [23, Lemma 1] and [14, Theorem 5]: there is $C>0$ such that

$$\mathcal{I}(f) \leqslant C\,\|f\|_{H^2_{\frac{d+1}{2}}} \qquad \forall\, f\in H^2_{\frac{d+1}{2}}.$$

We conclude then with Lemma B.3. □

With this result at hand, it remains to study the question about the behaviour of the Fisher information at $t=0$. To this end, we prove the following lemma.

Lemma 3.2. *Let $f=f(t,v)$ be a solution to (3.1) with initial datum f_0 with mass m_0, energy E_0 and entropy H_0 satisfying (1.7). Introduce for $i=1,\ldots,d$*

$$g_i = \partial_{v_i}\sqrt{f}\,,\quad a^i = \partial_{v_i}a\,,\quad b^i = \partial_{v_i}b\,,\quad g := \nabla\sqrt{f}\,.$$

Then, there exist A_0 and C_1 depending only on m_0, E_0, H_0 and the quantities (1.7) such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|g_i(t,v)|^2\mathrm{d}v + a_0\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\Big|\nabla g_i(t,v) - \frac{g_i(t,v)}{\sqrt{f(t,v)}}\,g(t,v)\Big|^2\mathrm{d}v \leqslant A_0\int_{\mathbb{R}^3}\langle v\rangle^{\gamma+2}\big|\nabla\sqrt{f(t,v)}\big|^2\mathrm{d}v + C_1\,. \tag{3.6}$$

Proof. With the notations of the lemma and recalling that $a = a(t,v)$ is symmetric, one can compute

$$\begin{aligned}-\partial_{v_i}\Big(\frac{1}{\sqrt{f}}\nabla\cdot(a\nabla f)\Big) &= -2\,\partial_{v_i}\Big(\frac{1}{\sqrt{f}}\nabla\cdot(ag\sqrt{f})\Big)\\ &= -2\,\nabla\cdot(a\nabla g_i) + 2\,\frac{g_i}{f}\,g\cdot ag - \frac{4}{\sqrt{f}}\,\nabla g_i\cdot ag - 2\nabla\cdot(a^i g) - \frac{2}{\sqrt{f}}\,g\cdot a^i g\,.\end{aligned}$$

We also have

$$\partial_{v_i}\Big(\frac{1}{\sqrt{f}}\nabla\cdot(bf)\Big) = \nabla\cdot(b\,g_i) + b\cdot\nabla g_i + \nabla\cdot(b^i\sqrt{f}) + b^i\cdot g\,.$$

As a consequence, after some integration by parts, the Dirichlet terms are computed as

$$\begin{aligned}\int_{\mathbb{R}^3}-\partial_{v_i}\Big(\frac{1}{\sqrt{f}}\nabla\cdot(a\nabla f)\Big)g_i\mathrm{d}v &= 2\int_{\mathbb{R}^3}\Big(a\nabla g_i\cdot\nabla g_i + \frac{g_i^2}{f}\,g\cdot ag - \frac{2g_i}{\sqrt{f}}\nabla g_i\cdot ag\Big)\mathrm{d}v\\ &\qquad + 2\int_{\mathbb{R}^3}\Big(a^i g\cdot\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\cdot a^i g\Big)\mathrm{d}v\\ &= 2\int_{\mathbb{R}^3}\Big|\sqrt{a}\Big(\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big)\Big|^2\mathrm{d}v + 2\int_{\mathbb{R}^3}a^i g\cdot\Big(\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big)\mathrm{d}v\,.\end{aligned}$$

Here $\sqrt{a} = \sqrt{a(t,v)}$ is the unique positive definite symmetric square root of $a(t,v)$. In addition,

$$\int_{\mathbb{R}^3}\partial_{v_i}\Big(\frac{1}{\sqrt{f}}\nabla\cdot(bf)\Big)\,g_i\mathrm{d}v = -\int_{\mathbb{R}^3}b^i\sqrt{f}\cdot\Big(\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big)\mathrm{d}v\,.$$

Consequently, we can find an energy estimate for g_i. Indeed, multiplying the Landau equation (3.1) by $1/\sqrt{f}$, differentiating in v_i, multiplying by g_i and integrating in velocity, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|g_i(t,v)|^2\mathrm{d}v &+ 2\int_{\mathbb{R}^3}\Big|\sqrt{a(t,v)}\Big(\nabla g_i(t,v) - \frac{g_i(t,v)}{\sqrt{f(t,v)}}\,g(t,v)\Big)\Big|^2\mathrm{d}v\\ &+ 2\int_{\mathbb{R}^3}a^i(t,v)g(t,v)\cdot\Big(\nabla g_i(t,v) - \frac{g_i(t,v)}{\sqrt{f(t,v)}}\,g(t,v)\Big)\mathrm{d}v\\ &- \int_{\mathbb{R}^3}\sqrt{f(t,v)}\,b^i(t,v)\cdot\Big(\nabla g_i(t,v) - \frac{g_i(t,v)}{\sqrt{f(t,v)}}\,g(t,v)\Big)\mathrm{d}v = 0\,.\end{aligned}$$

We proceed estimating each term, starting for the absorption term

$$\begin{aligned}\int_{\mathbb{R}^3}\Big|\sqrt{a}\Big(\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big)\Big|^2\mathrm{d}v &= \int_{\mathbb{R}^3}a\Big(\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big)\cdot\Big(\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big)\mathrm{d}v\\ &\geqslant a_0\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\Big|\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big|^2\mathrm{d}v\,.\end{aligned}$$

For the latter two terms we use Young's inequality $2|ab| \leqslant \epsilon a^2 + \epsilon^{-1}b^2$ with $\epsilon = 2\,a_0/3$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|g_i|^2\mathrm{d}v + a_0\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\Big|\nabla g_i - \frac{g_i}{\sqrt{f}}\,g\Big|^2\mathrm{d}v \leqslant \tfrac{3}{2a_0}\int_{\mathbb{R}^3}\langle v\rangle^{-\gamma}\big|a^i\,g\big|^2\mathrm{d}v + \tfrac{3}{4a_0}\int_{\mathbb{R}^3}\langle v\rangle^{-\gamma}\big|b^i\sqrt{f}\big|^2\mathrm{d}v\,.$$

We recall that $|b^i| \leqslant B(m_0,E_0)\langle v\rangle^{\gamma}$, therefore,

$$\int_{\mathbb{R}^3}\langle v\rangle^{-\gamma}\big|b^i\sqrt{f}\big|^2\mathrm{d}v \leqslant B(m_0,E_0)^2\int_{\mathbb{R}^3}\langle v\rangle^{\gamma}\,f\mathrm{d}v \leqslant C_1(m_0,E_0)\,.$$

Also, $|a^i| \leqslant A(m_0, E_0)\langle v\rangle^{\gamma+1}$. As a consequence,

$$\int_{\mathbb{R}^3} \langle v\rangle^{-\gamma}\big|a^i\, g\big|^2 \mathrm{d}v \leqslant A(m_0,E_0)\int_{\mathbb{R}^3}\langle v\rangle^{\gamma+2}|g|^2\mathrm{d}v = A\int_{\mathbb{R}^3}\langle v\rangle^{\gamma+2}\big|\nabla\sqrt{f}\big|^2\mathrm{d}v\,.$$

This gives the result. □

Proof of Theorem 1.2. For short time, say $t\in[0,1]$, integrate (3.6) in time and use Proposition 3.1 with $k=2$. Then, we can invoke Lemma 3.1 with $t_0=1$ to estimate $\mathcal{I}(f(t))$ for $t\geqslant 1$. □

3.1. **Exponential moments for the Landau equation.** In [14, Section 3] emergence and propagation of polynomial moments have been obtained for the Landau equation and, more recently [11, Section 3.2] develops the propagation of exponential moments for soft potentials. The starting point is the weak formulation for the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)\varphi(v)\,\mathrm{d}v = 2\sum_j\int_{\mathbb{R}^3} f(t,v)\,b_j\,\partial_{v_j}\varphi(v)\,\mathrm{d}v + \sum_{i,j}\int_{\mathbb{R}^3} f(t,v)\,a_{ij}\partial^2_{v_iv_j}\varphi(v)\,\mathrm{d}v. \tag{3.7}$$

Exponential moments can be easily studied in a similar fashion by choosing $\varphi(v)=e^{\lambda\langle v\rangle^s}$ with positive parameters λ, s to be determined. We note that, for such a choice,

$$\partial_{v_j}\varphi(v) = \lambda s\, e^{\lambda\langle v\rangle^s}\langle v\rangle^{s-2}v_j\,,\qquad \partial^2_{v_iv_j}\varphi(v) = \lambda s\,e^{\lambda\langle v\rangle^s}\Big((s-2)\langle v\rangle^{s-4}v_iv_j + \langle v\rangle^{s-2}\delta_{ij} + \lambda s\langle v\rangle^{2(s-2)}v_iv_j\Big)\,.$$

Thus, resuming the computations given in [14, pg. 201] one gets

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)e^{\lambda\langle v\rangle^s}\mathrm{d}v = \lambda s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & f(t,v)\,f(t,v_\star)|v-v_\star|^\gamma e^{\lambda\langle v\rangle^s}\langle v\rangle^{s-2}\\ &\times\Big(-2|v|^2+2|v_\star|^2+\big(|v|^2|v_\star|^2-(v\cdot v_\star)^2\big)\big((s-2)\langle v\rangle^{-2}+\lambda s\langle v\rangle^{s-2}\Big)\mathrm{d}v\,\mathrm{d}v_\star\,.\end{aligned}$$

At this point, we choose $0<s<2$ and thanks to the Young inequality $\lambda s\langle v\rangle^s\langle v_\star\rangle^2 \leqslant \frac{s}{2}\langle v\rangle + \frac{2-s}{2}(\lambda s)^{\frac{2}{2-s}}\langle v_\star\rangle^{\frac{4}{2-s}}$, we have

$$\begin{aligned}-2|v|^2+2|v_\star|^2+\big(|v|^2|v_\star|^2-(v\cdot v_\star)^2\big)\big((s-2)\langle v\rangle^{-2}+\lambda s\langle v\rangle^{s-2}\big) &\leqslant -2|v|^2+2|v_\star|^2+\lambda s\langle v\rangle^s|v_\star|^2\\ \leqslant -\frac{4-s}{2}\langle v\rangle^2+2\langle v_\star\rangle^2+\frac{(2-s)}{2}(\lambda s)^{\frac{2}{2-s}}\langle v_\star\rangle^{\frac{4}{2-s}} &\leqslant -\langle v\rangle^2+2\langle v_\star\rangle^2+C_s\lambda^{\frac{2}{2-s}}\langle v_\star\rangle^{\frac{4}{2-s}}\,.\end{aligned}$$

Thus, using Lemma B.2, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)e^{\lambda\langle v\rangle^s}\mathrm{d}v &\leqslant \lambda s\int_{\mathbb{R}^3} f(t,v)\,e^{\lambda\langle v\rangle^s}\langle v\rangle^{s+\gamma}\Big(-c+C\langle v\rangle^{-2}\Big)\mathrm{d}v\\ &\leqslant \lambda s\int_{\mathbb{R}^3} f(t,v)\,e^{\lambda\langle v\rangle^s}\langle v\rangle^{s+\gamma}\Big(-\frac{c}{2}+C\,\mathbf{1}_{\{|v|\leqslant r\}}\Big)\mathrm{d}v\end{aligned}\tag{3.8}$$

where $c>0$ depends on m_0, E_0. Meanwhile,

$$C = 2\sup_{t\geqslant 0}\|f(t)\|_{L^1_{2+\gamma}} + C_s\lambda^{\frac{2}{2-s}}\sup_{t\geqslant0}\|f(t)\|_{L^1_{\frac{4}{2-s}+\gamma}}\,,\qquad\text{and}\qquad r:=r(C,c,\gamma)\,.$$

This proves a propagation result for exponential moments.

Proposition 3.2. *Fix $s\in(0,\gamma]$ and assume that f_0 belongs to $L^1_{2+\gamma}(\mathbb{R}^3)\cap L^1_{\log}(\mathbb{R}^3)$. Then, for the solution $f(t,v)$ of the Landau equation with initial datum f_0 given by* [14, Theorem 5] *there exists some $\beta:=\beta_{s,\gamma}\geqslant 1$ such that*

$$\sup_{t\geqslant0}\int_{\mathbb{R}^3} f(t,v)e^{\min\{1,t^\beta\}\langle v\rangle^s}\mathrm{d}v\leqslant C(f_0)\qquad\textit{(Emergence of tails)}.$$

Fix $s\in(0,2)$, $\lambda>0$, and assume that $\int_{\mathbb{R}^3} f_0\,e^{\lambda\langle v\rangle^s}\mathrm{d}v<\infty$. Then, for the solution $f(t,v)$ of the Landau equation with initial datum f_0 given by [14, Theorem 5] *it follows that*

$$\sup_{t\geqslant0}\int_{\mathbb{R}^3} f(t,v)e^{\lambda\langle v\rangle^s}\mathrm{d}v\leqslant C_{\lambda,s}(f_0)\qquad\textit{(Propagation of tails)}.$$

Proof. For the emergence of the exponential tail we assume $t \in (0,1)$ and take $\varphi(t,v) = e^{t^{\beta}\langle v\rangle^{s}}$ with $s \in (0,2)$ and $\beta > 0$ to be chosen. We repeat the steps leading to estimate (3.8) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)e^{t^{\beta}\langle v\rangle^{s}}\mathrm{d}v \leqslant t^{\beta}s\int_{\mathbb{R}^3} f(t,v)\,e^{t^{\beta}\langle v\rangle^{s}}\langle v\rangle^{s+\gamma}\Big(-c + C(t)\,\langle v\rangle^{-2} + \frac{\beta}{st}\langle v\rangle^{-\gamma}\Big)\mathrm{d}v\,. \tag{3.9}$$

The constant $c > 0$ depends on m_0, E_0 whereas $C(t)$ is given by

$$C(t) = 2\,\|f(t)\|_{L^1_{2+\gamma}} + C_s t^{\frac{2\beta}{2-s}}\|f(t)\|_{L^1_{\frac{4}{2-s}+\gamma}}\,.$$

Similarly to the Boltzmann equation, one can prove with the techniques given in [14, Section 3] that $\|f\|_{L^1_k} \lesssim t^{-k/\gamma}$. Therefore, choosing

$$\beta = \frac{4+(2-s)\gamma}{(4-s)\gamma} > 1\,,$$

we guarantee that $C(t) \lesssim t^{-\beta}$. Thus,

$$-c + C(t)\langle v\rangle^{-2} + \frac{\beta}{st}\langle v\rangle^{-\gamma} \leqslant -c + \frac{C_1}{t^{\beta}}\langle v\rangle^{-\gamma} \leqslant -\frac{c}{2} + \frac{C_1}{t^{\beta}}\,\mathbf{1}_{\{|v|\leqslant t^{-\beta/\gamma}r\}}\,,$$

where the radius $r := r(C_1,c)$ is independent of time. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)e^{t^{\beta}\langle v\rangle^{s}}\mathrm{d}v \leqslant s\,C_1\,e^{t^{\beta(1-s/\gamma)}\langle r\rangle^{s}}\int_{\mathbb{R}^3} f(t,v)\,\langle v\rangle^{s+\gamma}\mathrm{d}v \leqslant \tilde{C}(f_0)\,, \quad 0 < s \leqslant \gamma\,.$$

This proves the generation of the exponential tail. □

As previously expressed for the Boltzmann equation, the propagation/generation of the Fisher information and the exponential moments imply the propagation/generation of the exponential moments for the gradient of solutions. For any $s \in (0,\gamma]$

$$\begin{aligned}\int_{\mathbb{R}^3}\big|\nabla f(t,v)\big|e^{\frac{\min\{1,t^{\beta}\}}{2}\langle v\rangle^{s}}\mathrm{d}v &= 2\int_{\mathbb{R}^3}\big|\nabla\sqrt{f}\big|\sqrt{f}\,e^{\frac{\min\{1,t^{\beta}\}}{2}\langle v\rangle^{s}}\mathrm{d}v\\ &\leqslant \mathcal{I}(f(t))^{\frac12}\big\|f(t)e^{\min\{1,t^{\beta}\}\langle v\rangle^{s}}\big\|_{L^1}^{\frac12} \leqslant C(f_0)\,.\end{aligned}$$

Appendix A. Regularity estimates for the Boltzmann equation

We include here some classical results in the theory of the homogeneous Boltzmann equation. We use them in the core of this note.

Theorem A.1. *Let $b \in L^1(\mathbb{S}^{d-1})$ be the scattering kernel and $\gamma \in (0,1]$. Let $0 \leqslant f_0 \in L^1_2(\mathbb{R}^d) \cap L^1_{log}(\mathbb{R}^d)$ be the initial data. Then, the unique solution to (1.3) satisfies: for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|\log f(t,v)| \leqslant C_\varepsilon\left(1+\log^{+}(1/t)\right)\langle v\rangle^{2+\varepsilon} + f(t,v), \qquad v \in \mathbb{R}^d, \qquad t > 0.$$

Proof. The proof relies on [20, Theorem 1.1 & Lemma 3.1] and follows after keeping track of the time dependence of the constants involved. A similar argument was made to prove [1, Theorem 3.5]. □

Theorem A.2. *(See [28, Theorem 4.2] and [2, Lemma 8]) Let $b \in L^1(\mathbb{S}^{d-1})$ be the scattering kernel, $\gamma \in (0,1]$, and assume $0 \leqslant f_0 \in L^1_2(\mathbb{R}^d)$. Then, for every $k > 0$ there exists a constant $C_k \geqslant 0$ depending only on k, b, and the initial mass and energy of f_0, such that*

$$m_k(t) := \int_{\mathbb{R}^d} f(t,v)|v|^k\mathrm{d}v \leqslant C_k\max(1,t^{-k/\gamma}) \qquad \text{for } t > 0.$$

If, in addition, $m_k(0) < \infty$ then

$$\sup_{t\geqslant 0} m_k(t) \leqslant C_k\,,$$

for some constant C_k depending only on k, b, the mass and energy of f_0, and $m_k(0)$.

Lemma A.1. *Let $b \in L^1(\mathbb{S}^{d-1})$ be the scattering kernel and $\gamma \in (0,1]$. Let $0 \leqslant f(t) \in L^1_{2+\varepsilon}(\mathbb{R}^d)$, with $\varepsilon > 0$, be such that for some $C \geqslant c > 0$*

$$C \geqslant \int_{\mathbb{R}^d} f(t,v)\langle v\rangle^2 \mathrm{d}v \geqslant c, \qquad \int_{\mathbb{R}^d} f(t,v)\, v\, \mathrm{d}v = 0\,.$$

Then, there exists κ_0 depending on C, c, b and $\sup_{t\geqslant 0}\|f(t)\|_{L^1_{2+\varepsilon}}$ such that

$$\mathcal{R}(f)(v) \geqslant \kappa_0 \langle v\rangle^{\gamma}. \tag{A.1}$$

Moreover,

$$0 \leqslant \Delta_v \mathcal{R}(f)(v) \leqslant C_{d,\gamma}\|b\|_{L^1(\mathbb{S}^{d-1})}\Big(\|f\|_{L^1} + \|f\|_{H^{\frac{(4-d)+}{2}}}\Big). \tag{A.2}$$

Proof. The lower bound (A.1) has been established in [5, Lemma 2.1]. Let us focus on the second point by directly computing

$$\Delta_v \mathcal{R}(f)(v) = \mathrm{div}_v\left(\nabla \mathcal{R}(f)(v)\right) = \gamma\,\|b\|_{L^1(\mathbb{S}^{d-1})}\int_{\mathbb{R}^d} \mathrm{div}_v\left((v-v_\star)|v-v_\star|^{\gamma-2}\right) f(v_\star)\mathrm{d}v_\star.$$

Since $\mathrm{div}_v\left((v-v_\star)|v-v_\star|^{\gamma-2}\right) = (d+\gamma-2)|v-v_\star|^{\gamma-2}$, we get

$$\begin{aligned}
0 \leqslant \Delta_v \mathcal{R}(f)(v) &= \gamma\,(d+\gamma-2)\|b\|_{L^1(\mathbb{S}^{d-1})}\int_{\mathbb{R}^d} |v-v_\star|^{\gamma-2} f(v_\star)\mathrm{d}v_\star \\
&\leqslant \gamma\,(d+\gamma-2)\|b\|_{L^1(\mathbb{S}^{d-1})}\left(\left(\int_{\mathbb{R}^d}\big|f(v_\star)\big|^{\frac{d}{d-2}}\mathrm{d}v_\star\right)^{\frac{d-2}{d}}\left(\int_{\{|v_\star|\leqslant 1\}}\big|v_\star\big|^{\frac{d(\gamma-2)}{2}}\mathrm{d}v_\star\right)^{\frac{2}{d}} + \int_{\mathbb{R}^d} f(v_\star)\mathrm{d}v_\star\right) \\
&\leqslant C_{d,\gamma}\Big(\|f\|_{L^1} + \|f\|_{H^{\frac{(4-d)+}{2}}}\Big)\,.
\end{aligned}$$

For the last inequality we used the Sobolev embedding valid for $d \geqslant 3$. $\square$

Theorem A.3. *(See [3, Corollary 1.1] and [19, Theorem 4.1]) Let $b \in L^1(\mathbb{S}^{d-1})$ be the scattering kernel and $\gamma \in (0,1]$. For a fixed $\eta \geqslant 0$ assume that*

$$0 \leqslant f_0 \in L^1_{\eta+d}(\mathbb{R}^d) \cap L^2_\eta(\mathbb{R}^d)\,.$$

Then,

$$\sup_{t\geqslant 0}\|f(t)\|_{L^2_\eta} < \infty.$$

Theorem A.4. *(See [7, Theorem 2.1] and [19, Theorem 3.5]) Let $b \in L^2(\mathbb{S}^{d-1})$ be the scattering kernel and $\gamma \in (0,1]$. Then, for all $s \geqslant 0$ and all $\eta \geqslant 0$, it holds*

$$\|\mathcal{Q}^+(g,f)\|_{H_\eta^{s+\frac{d-1}{2}}} \leqslant C_d\left(\|g\|_{H^s_{\eta+1+\gamma}}\|f\|_{H^s_{\eta+1+\gamma}} + \|g\|_{L^1_{\eta+\gamma}}\|f\|_{L^1_{\eta+\gamma}}\right).$$

for some positive constant C_d depending only on the dimension d.

Theorem A.5. *(See [19, Theorem 4.2]) Let $b \in L^2(\mathbb{S}^{d-1})$ be the scattering kernel and $\gamma \in (0,1]$. Let $\eta \geqslant 0$ and assume that the initial datum f_0 satisfies*

$$f_0 \in L^1_{\eta+1+\gamma/2+d}(\mathbb{R}^d) \cap L^2_{\eta+1+\gamma/2}(\mathbb{R}^d) \cap H^1_\eta(\mathbb{R}^d)\,.$$

Then, the unique solution $f(t,v)$ to (1.3) with initial condition f_0 satisfies

$$\sup_{t\geqslant 0}\|f(t)\|_{H^1_\eta} := C_\eta < \infty\,.$$

Proof. Set $g(t,v) = \nabla f(t,v)$ so that $\partial_t g(t,v) = \nabla \mathcal{Q}(f,f)(t,v)$. Applying the inner product of such equation with $\langle v\rangle^{2\eta} g(t,v)$ and integrating over $\mathbb{R}^d$ we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{L^2_\eta} = \int_{\mathbb{R}^d}\langle v\rangle^{2\eta} g(t,v)\cdot\nabla\mathcal{Q}^+(f,f)(t,v)\mathrm{d}v - \int_{\mathbb{R}^d}\langle v\rangle^{2\eta} g(t,v)\cdot\nabla\mathcal{Q}^-(f,f)(t,v)\mathrm{d}v.$$

Notice that

$$\nabla \mathcal{Q}^{-}(f,f)(t,v) = g(t,v)\mathcal{R}(f(t,\cdot))(v) + f(t,v)\nabla\mathcal{R}(f(t,\cdot))(v)$$

so that, after using (A.1),

$$\int_{\mathbb{R}^d} \langle v\rangle^{2\eta} g(t,v)\cdot\nabla\mathcal{Q}^{-}(f,f)(t,v)\mathrm{d}v \geqslant \kappa_0\|g(t)\|^2_{L^2_{\eta+\gamma/2}} + \int_{\mathbb{R}^d}\langle v\rangle^{2\eta} f(t,v)g(t,v)\cdot\nabla\mathcal{R}(f)(t,v)\mathrm{d}v.$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{L^2_\eta} + \kappa_0\|g(t)\|^2_{L^2_{\eta+\gamma/2}} \leqslant \|g(t)\|_{L^2_{\eta+\gamma/2}}&\|\nabla\mathcal{Q}^{+}(f(t),f(t))\|_{L^2_{\eta-\gamma/2}}\\ &- \int_{\mathbb{R}^d}\langle v\rangle^{2\eta} f(t,v)g(t,v)\cdot\nabla\mathcal{R}(f)(t,v)\mathrm{d}v\,.\end{aligned}\tag{A.3}$$

Since

$$\left|\nabla\mathcal{R}(f)\right| \leqslant \gamma\|b\|_{L^1(\mathbb{S}^{d-1})}C_d\Big(\|f\|_{L^1}+\|f\|_{L^2}\Big)\,,$$

we estimate this last integral as

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\langle v\rangle^{2\eta} f(t,v)g(t,v)\cdot\nabla\mathcal{R}(f)(t,v)\mathrm{d}v\right| &\leqslant C(f_0)\|b\|_{L^1(\mathbb{S}^{d-1})}\int_{\mathbb{R}^d}\langle v\rangle^{2\eta}|g(t,v)|\,f(t,v)\mathrm{d}v\\ &\leqslant C(f_0)\|b\|_{L^1(\mathbb{S}^{d-1})}\|f(t)\|_{L^2_\eta}\|g(t)\|_{L^2_\eta} \leqslant C(f_0,b)\|g(t)\|_{L^2_\eta}\,.\end{aligned}\tag{A.4}$$

Using (A.4) and Theorem A.4 in (A.3), we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{L^2_\eta} + \kappa_0\|g(t)\|^2_{L^2_{\eta+\gamma/2}} \leqslant C_3\|g(t)\|_{L^2_{\eta+\gamma/2}}\Big(\|f(t)\|^2_{L^1_{\eta+\gamma/2}} + \|f(t)\|^2_{L^2_{\eta+1+\gamma/2}}\Big) + C(f_0,b)\|g(t)\|_{L^2_\eta}\,.$$

Thus, since

$$\sup_{t\geqslant 0}\Big(\|f(t)\|_{L^2_{\eta+1+\gamma/2}} + \|f(t)\|_{L^1_{\eta+\gamma/2}}\Big) \leqslant C(f_0)$$

according to Theorems A.2 and A.3 and our hypothesis on f_0, it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{L^2_\eta} + \kappa_0\|g(t)\|^2_{L^2_{\eta+\gamma/2}} \leqslant C(f_0,b)\,\|g(t)\|_{L^2_{\eta+\gamma/2}}, \qquad \forall t>0\,,$$

which readily gives that

$$\sup_{t\geqslant 0}\|g(t)\|_{L^2_\eta} \leqslant \max\left\{\|g_0\|_{L^2_\eta}, \tfrac{C(f_0,b)}{\kappa_0}\right\}.$$

This together with the propagation of $\|f\|_{L^2_\eta}$ proves the result. □

Appendix B. Regularity estimates for the Landau equation

We collect here known results, extracted from [14] about the regularity of solutions to the Landau equation (1.5). We begin with classical estimate related to the matrix $A(z)$. For $(i,j)\in[\![1,3]\!]^2$, we recall that

$$A(z) = \left(A_{i,j}(z)\right)_{i,j} \quad \text{with} \quad A_{i,j}(z) = |z|^{\gamma+2}\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right),$$

and introduce

$$B_i(z) = \sum_k \partial_k A_{i,k}(z) = -2\,z_i\,|z|^\gamma.$$

For any $f\in L^1_{2+\gamma}(\mathbb{R}^3)$, we define then the matrix-valued mapping $a(v)=A*f(v)$ and the vector-valued mapping $b(v)=(b_i(v))_i$ with

$$b_i(v) = B_i * f, \qquad \forall v\in\mathbb{R}^3, \qquad i=1,\ldots,3.$$

One has the following [14, Proposition 4].

Lemma B.1. *For any nonnegative function $f \in L^1_2 \cap L^1_{\log}(\mathbb{R}^3)$ satisfying*

$$\int_{\mathbb{R}^3} f(v)\mathrm{d}v = m_0, \quad \int_{\mathbb{R}^3} |v|^2 f(v)\mathrm{d}v \leqslant E_0\,, \quad \text{and} \quad \int_{\mathbb{R}^3} f(v)\log f(v)\mathrm{d}v \leqslant H_0\,,$$

there is a positive constant a_0 depending only on m_0, E_0, and H_0 such that

$$a(v)\,\xi\cdot\xi = \sum_{i,j=1}^{3} a_{ij}(v)\xi_i\xi_j \geqslant a_0\,\langle v\rangle^{\gamma}\,|\xi|^2\,, \qquad \forall\, v\in\mathbb{R}^3,\;\; \xi\in\mathbb{R}^3\,. \tag{B.1}$$

Also, if $f \in L^1_{\gamma+2}(\mathbb{R}^3)$, there exists a positive constant $C>0$ depending on $\|f\|_{L^1_{\gamma+2}}$ such that

$$a(v)\,\xi\cdot\xi \leqslant C\langle v\rangle^{\gamma+2}|\xi|^2 \qquad \forall \xi\in\mathbb{R}^3,\; v\in\mathbb{R}^3.$$

Remark B.1. *Note that*

$$\begin{cases} \;\;\big|\,b(v)\,\big| & \leqslant 2\langle v\rangle^{\gamma+1}\|f\|_{L^1_{\gamma+1}} \leqslant 2\langle v\rangle^{\gamma+1}\|f\|_{L^1_2}\,, \\ \big|\,\nabla\cdot b(v)\,\big| & \leqslant 8\langle v\rangle^{\gamma}\|f\|_{L^1_{\gamma}} \leqslant 8\langle v\rangle^{\gamma}\|f\|_{L^1_2}\,, \end{cases} \tag{B.2}$$

since $0\leqslant\gamma\leqslant 1$.

Here, $f(t,v)$ will denote a weak solution to (1.5) associated to an initial datum f_0 with mass m_0, energy E_0 and entropy H_0. One has then the following result about propagation and appearance of moments, see [14, Theorem 3].

Lemma B.2. *For any $s\geqslant 0$,*

$$\int_{\mathbb{R}^3} \langle v\rangle^s f_0(v)\mathrm{d}v < \infty \quad \Longrightarrow \quad \sup_{t\geqslant 0}\int_{\mathbb{R}^3}\langle v\rangle^s f(t,v)\mathrm{d}v < \infty.$$

Moreover, for any $t_0>0$ and any $s>0$ there exists $C>0$ depending only on m_0, E_0, H_0, s and t_0 such that

$$\sup_{t\geqslant t_0}\int_{\mathbb{R}^3}\langle v\rangle^s f(t,v)\mathrm{d}v \leqslant C.$$

We have then the following result about instantaneous appearance and uniform bounds for regularity, see [14, Theorem 5].

Lemma B.3. *For any $t_0>0$, any integer $k\in\mathbb{N}$ and $s>0$, there exists a constant $C_{t_0}>0$ depending only on m_0, E_0, H_0, k, s and $t_0>0$ such that*

$$\sup_{t\geqslant t_0}\|f(t)\|_{H^k_s} \leqslant C_{t_0}.$$

We end this section with a simple estimate for integral of the type

$$\int_{\mathbb{R}^d}\langle v\rangle^k f(v)\,|\log f(v)|\,\mathrm{d}v, \qquad k\geqslant 0\,,$$

yielding to estimate (3.5). Set, for notational simplicity,

$$m_k := \int_{\mathbb{R}^d}\langle v\rangle^k\, f(v)\,\mathrm{d}v, \qquad k\geqslant 0\,.$$

Let us emphasize that, contrary to the previous results of this appendix, in the following lemma, the dimension $d\geqslant 2$ is arbitrary and the function f is not restricted to a solution to the Landau equation.

Lemma B.4. *For any $k\geqslant 0$ and any $\varepsilon>0$, there exists $C_k(\varepsilon)>0$ such that, for any nonnegative $f\in L^1_{k+\varepsilon}(\mathbb{R}^d)$, one has*

$$\int_{\mathbb{R}^d}\langle v\rangle^k f(v)\,|\log f(v)|\,\mathrm{d}v \leqslant \int_{\mathbb{R}^d}\langle v\rangle^k f(v)\log f(v)\mathrm{d}v + 2m_{k+\varepsilon} + C_k(\varepsilon)\,. \tag{B.3}$$

Furthermore, for any $k \geqslant 0$, $\delta > 0$ and any $\varepsilon > 0$, there exist $K_k(\delta)$ and $C_k(\varepsilon)$ such that, for any nonnegative $f \in L^1_{k+\varepsilon}(\mathbb{R}^d)$, one has

$$\int_{\mathbb{R}^d} \langle v\rangle^k f(v)\,|\log f(v)|\,\mathrm{d}v \;\leqslant\; \delta \int_{\mathbb{R}^d} \langle v\rangle^k \left|\nabla\sqrt{f}\right|^2 \mathrm{d}v \,+\, K_k(\delta)(1+|\log m_k|)m_k + 2m_{k+\varepsilon} + C_k(\varepsilon)\,. \qquad \text{(B.4)}$$

Proof. Given $k \geqslant 0$, we denote by

$$\mathcal{H}_k(f) = \int_{\mathbb{R}^d} f(v)\log f(v)\langle v\rangle^k \mathrm{d}v, \qquad \mathbf{H}_k(f) = \int_{\mathbb{R}^d} \langle v\rangle^k f(v)|\log f(v)|\mathrm{d}v\,.$$

We set $A = \{v \in \mathbb{R}^d\,,\ f(v) < 1\}$, $A^c = \{v \in \mathbb{R}^d\,,\ f(v) \geqslant 1\}$ so that

$$\begin{aligned}\mathbf{H}_k(f) &= \int_{A^c} f(v)\log f(v)\langle v\rangle^k \mathrm{d}v - \int_A f(v)\log f(v)\langle v\rangle^k\mathrm{d}v = \mathcal{H}_k(f) - 2\int_A f(v)\log f(v)\langle v\rangle^k \mathrm{d}v\\ &= \mathcal{H}_k(f) + 2\int_A f(v)\log\left(\frac{1}{f(v)}\right)\langle v\rangle^k\mathrm{d}v\,.\end{aligned}$$

Given $\varepsilon > 0$, set now $B = \{v \in \mathbb{R}^d\,;\ f(v) \geqslant \exp(-\langle v\rangle^\varepsilon)\}$. If $v \in A\cap B$, then $\log(\frac{1}{f(v)}) \leqslant \langle v\rangle^\varepsilon$ and

$$\mathbf{H}_k(f) \leqslant \mathcal{H}_k(f) + 2m_{k+\varepsilon} + 2\int_{A\cap B^c} f(v)\log\left(\frac{1}{f(v)}\right)\langle v\rangle^k\mathrm{d}v.$$

Now, since $x\log(1/x) \leqslant \frac{2}{e}\sqrt{x}$ for any $x \in (0,1)$, we get

$$\int_{A\cap B^c} f(v)\log\left(\frac{1}{f(v)}\right)\langle v\rangle^k\mathrm{d}v \leqslant \frac{2}{e}\int_{\mathbb{R}^d}\exp\left(-\frac{\langle v\rangle^\varepsilon}{2}\right)\langle v\rangle^k\mathrm{d}v =: C_k(\varepsilon) < \infty,$$

which gives (B.3). Now, setting $g^2(v) = \langle v\rangle^k f(v)$, one sees that

$$\mathcal{H}_k(f) = \int_{\mathbb{R}^d} g^2(v)\log g^2(v)\mathrm{d}v - k\int_{\mathbb{R}^d} g^2(v)\log\langle v\rangle\mathrm{d}v \leqslant \int_{\mathbb{R}^d} g^2(v)\log g^2(v)\mathrm{d}v$$

since $\langle v\rangle \geqslant 1$. We can invoke now the Euclidian logarithmic Sobolev inequality [18, Theorem 8.14]

$$\int_{\mathbb{R}^d} g^2\log\frac{g^2}{\|g\|^2_{L^2}}\mathrm{d}v + d\big(1+\tfrac{1}{2}\log\delta\big)\|g\|^2_{L^2} \leqslant \frac{\delta}{\pi}\int_{\mathbb{R}^d}|\nabla g|^2\,\mathrm{d}v, \qquad \forall\,\delta > 0$$

to obtain, observe that $\|g\|^2_{L^2} = m_k$, that

$$\mathcal{H}_k(f) \leqslant \frac{\delta}{\pi}\int_{\mathbb{R}^d}|\nabla g|^2\,\mathrm{d}v + m_k\log m_k - d\big(1+\tfrac{1}{2}\log\delta\big)m_k, \qquad \forall\delta > 0\,.$$

Furthermore, there exists $C_k > 0$ such that

$$\int_{\mathbb{R}^d}|\nabla g|^2\,\mathrm{d}v = \int_{\mathbb{R}^d}\left|\nabla\left(\langle v\rangle^{\frac{k}{2}}\sqrt{f(v)}\right)\right|^2\mathrm{d}v \leqslant C_k\left(\int_{\mathbb{R}^d}\langle v\rangle^k|\nabla\sqrt{f}|^2\mathrm{d}v + m_k\right)$$

from which we get the result. □

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